CONTINUUM-DISCRETE MODEL OF A MIXTURE OF GAS AND SOLID PARTICLES FOR SMALL CONCENTRATION OF PARTICLES

S. P. Kiselev and V. M. Fomin

UDC 532.529+533.6.011

It was noted in [1-3] that one of the deficiencies of the two-fluid model is the possibility of intersection of the particle trajectories and as a result an infinite mean density of particles at the intersection points. It was suggested in [2] that for flows with Kn << 1 there should be clusters of densely packed particles at the intersection points of the particle trajectories due to particle collisions. In the present paper we consider the case Kn >> 1. The ensemble of particles is described with the help of the collisionless kinetic equation. It is shown that in this model intersections of the particle trajectories are possible, and that the mean particle density remains finite everywhere. We study the stability of the flow of the mixture of gas and particles against small perturbations in the framework of our model. Unlike the two-fluid model [1, 2, 4] the perturbations are bounded and their amplitude is inversely proportional to the width of the velocity distribution function of the particles raised to a fractional power. The finite amplitude of the perturbations in [2] results from collisions in the particle "gas."

1. The kinetic equation for the particles in the collisionless limit can be found from the equations of [5]. Neglecting collisions and the diffusion of particles in velocity space (small m_2), it has the form

$$\frac{\partial \mathcal{F}}{\partial t} + \mathbf{u}_2 \frac{\partial \mathcal{F}}{\partial \mathbf{r}} + \left(\mathbf{g} - \frac{1}{\rho_{22}} \nabla p\right) \frac{\partial \mathcal{F}}{\partial \mathbf{u}_2} + \frac{\partial}{\partial \mathbf{u}_2}\left[\frac{(\mathbf{u}_1 - \mathbf{u}_2)}{\tau} \mathcal{F}\right] = 0, \tag{1.1}$$

where $\mathbf{u}_1(t, \mathbf{r})$ and $p(t, \mathbf{r})$ are the velocity and pressure of the gas; $\mathbf{u}_2$ and ρ_{22} are the velocity and true density of the particles, $\mathcal{F}(t, \mathbf{r}, \mathbf{u}_2)$ is the single-particle distribution function of the particles, $\mathbf{g}$ is the acceleration of gravity, τ is the relaxation time, and m_2 is the volume concentration of particles.

The above simplifications occur when the following inequality is satisfied:

$$\mathrm{Kn} \simeq d/m_2 L \gg 1. \tag{1.2}$$

Here d is the diameter of a particle and L is the characteristic length of the variation of the mean flow parameters. Equations for the mean quantities are determined as in [5]

$$m_2 = \int_{-\infty}^{+\infty} \mathcal{F} d\mathbf{u}_2, \ \langle \mathbf{u}_2 \rangle = \frac{1}{m_2} \int_{-\infty}^{+\infty} \mathbf{u}_2 \mathcal{F} d\mathbf{u}_2. \tag{1.3}$$

Neglecting the effects of heat exchange between the gas and the particles, the system (1.1)-(1.3) is closed by the equations for the gas phase obtained in [6]:

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial}{\partial \mathbf{r}}(\rho_1 \mathbf{u}_1) = 0, \ \rho_1 = \rho_{11} m_1, \ m_1 + m_2 = 1, \tag{1.4}$$
$$\rho_1 \frac{d_1 \mathbf{u}_1}{dt} = \rho_1 \mathbf{g} - m_1 \nabla p - \rho_2 \left(\frac{\mathbf{u}_1 - \langle \mathbf{u}_2 \rangle}{\tau}\right), \ \frac{d_1}{dt} = \frac{\partial}{\partial t} + \mathbf{u}_1 \frac{\partial}{\partial \mathbf{r}},$$
$$p = p(\rho_{11}), \ \rho_{22} = \mathrm{const}, \ \rho_2 = \rho_{22} m_2,$$

where ρ_{11} is the true density of the gas, m_1 is the volume concentration of the gas, and $\langle u_2 \rangle$ is the average velocity of the particles.

Equation (1.1) is represented in the form

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 93-101, March-April, 1986. Original article submitted July 14, 1984.

0021-8944/86/2702-0242$12.50

$$D\mathcal{F}/Dt = -\mathcal{F}\partial \mathbf{F}'/\partial \mathbf{u}_2,$$
$$D/Dt = \partial/\partial t + \mathbf{u}_2\partial/\partial \mathbf{r} + \mathbf{F}\partial/\partial \mathbf{u}_2,\ \mathbf{F}' = (\mathbf{u}_1 - \mathbf{u}_2)/\tau,$$

and its general solution is written in the form

$$\mathcal{F}(t, \mathbf{r}, \mathbf{u}_2) = \mathcal{F}^0(t^0, \mathbf{r}^*, \mathbf{u}_2^*) \exp\left(-\int_{t^0}^{t} \frac{\partial \mathbf{F}'}{\partial \mathbf{u}_2} dt\right), \tag{1.5}$$
$$\mathbf{r} = \mathbf{r}(t, t^0, \mathbf{r}_2^*, \mathbf{u}_2^*),\ \mathbf{u}_2 = \mathbf{u}_2(t, t^0, \mathbf{r}_2^*, \mathbf{u}_2^*),$$
$$\mathbf{u}_2^* = \mathbf{u}_2|_{t=t^0},\ \mathbf{r}_2^* = \mathbf{r}_2|_{t=t^0}.$$

Here the integral is taken along the characteristics determined by the second and third equations in (1.5), and $\mathcal{F}^0$ is an arbitrary function. Multiplying (1.1) by 1 and u_2, respectively, and integrating with respect to u_2 from $-\infty$ to $+\infty$, we find, in the special case where the particle trajectories do not intersect in the region of flow and random motion near the particles is absent at the initial instant of time, the solution of the system (1.1)-(1.4) reduces to the solution obtained with the two-fluid model.

2. We consider a one-dimensional problem on the decay of a discontinuity in the mixture of gas and solid particles in the region $D_Z\{-\infty < x_2 < +\infty,\ t \geqq 0\}$. The volume concentration of particles m_2 will be assumed to be so small that it is possible to neglect the effect of the particles on the gas. Assuming $u_1 = \text{const}$, $p = \text{const}$, $\rho_{11} = \text{const}$, $g = 0$, we transform the system (1.1)-(1.3) to the form

$$\frac{\partial \mathcal{F}}{\partial t} + u_2 \frac{\partial \mathcal{F}}{\partial x_2} + \frac{\partial}{\partial u_2}\left(\frac{u_1 - u_2}{\tau}\mathcal{F}\right) = 0, \tag{2.1}$$
$$m_2 = \int_{-\infty}^{+\infty} \mathcal{F} du_2,\ \langle u_2\rangle = \frac{1}{m_2}\int_{-\infty}^{+\infty} u_2 \mathcal{F} du_2.$$

For system (2.1) in the region D_Z we have the following initial conditions:

$$\mathcal{F}^0(t = 0) = \frac{m_2^0}{\sqrt{2\pi\sigma}} \exp\left(-\frac{(u_2 - u_2^0(x_2))^2}{2\sigma}\right), \tag{2.2}$$
$$u_2^0(x_2) = w^s - w^\pi \operatorname{arctg} \alpha x_2,$$
$$w^s = (w_1 + w_2)/2,\ w^\pi = (w_1 - w_2)/\pi,$$
$$w_1 > w_2 > u_1 > 0,$$

where $u_2{}^0(x_2)$ is a "smeared" step function with width $\Delta x_2 \sim 1/\alpha$ and $u_2{}^0 \to w_1$ when $x_2 \to -\infty$ and $u_2{}^0 \to w_2$ when $x_2 \to +\infty$. The characteristics of Eq. (2.1) are determined from the equations

$$dx_2/dt = u_2,\ du_2/dt = (u_1 - u_2)/\tau. \tag{2.3}$$

Introducing the notation $u_2{}^* = u_2(t = 0)$, $x_2{}^* = x_2(t = 0)$, we obtain from (2.3)

$$u_2 = u_1 + (u_2^* - u_1)\,e^{-t/\tau},$$
$$x_2 = x_2^* + u_1 t + (u_2^* - u_1)K,\ K = \tau(1 - e^{-t/\tau}).$$

Substituting (2.2) in (1.5) and introducing a change of variables

$$du_2 = \frac{\partial u_2}{\partial u_2^*} du_2^* = e^{-t/\tau} du_2^*,$$

we find

$$m_2 = \frac{m_2^0}{\sqrt{2\pi\sigma}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(u_2^* - u_2^0(x_2^*))^2}{2\sigma}\right) du_2^*. \tag{2.4}$$

The integral (2.4) is evaluated by the saddle-point method for fixed values of t and x_2.

The function $\varphi(u_2^*, t, x_2) = (u_2^* - w^s + w^\pi \arctan \alpha x_2^*)^2$ is expanded in a Taylor series up to terms of order $(y - u_2^*)^2$, where y and x_2^* are determined from the system of equations

$$y - w^s + w^\pi \operatorname{arctg} \alpha x_2^* = 0, \; x_2^* + u_1 t + (y - u_1) K - x_2 = 0. \tag{2.5}$$

Substituting (2.5) into (2.4) we have

$$m_2 = \sum_{l=1}^{n} m_2^0 \left| \left| 1 - \frac{w^\pi \alpha K}{1 + (\alpha x_2^*)^2} \right|, \; x_2 = x_2^* + u_1 (t - K) + u_2^0 (x_2^*) K. \right. \tag{2.6}$$

Here t and x_2 are fixed and n is the number of roots of the second equation of (2.6), which is solved graphically (Fig. 1), where $Y = w^\pi \arctan \alpha x_2^* - w^s$, $X = (x_2^* - x_2 + u_1(t - K))/K$. The root is given by the intersection point of the straight line X and the curve Y, i.e., (Y = X). It follows from the results shown in Fig. 1 that in the region bounded by the lines passing through the points (ν^-, ξ^+) and (ν^+, ξ^-), the number of roots is n = 3 and elsewhere n = 1. Differentiating the second equation of (2.6), we obtain

$$\left(\frac{\partial x_2}{\partial x_2^*}\right)_t = 1 - \frac{\alpha w^\pi K}{1 + (\alpha x_2^*)^2}, \text{ hence} \tag{2.7}$$

$$\frac{\partial x_2}{\partial x_2^*} > 0, \quad -\infty < x_2^* < \xi^-(t),$$

$$\frac{\partial x_2}{\partial x_2^*} < 0, \quad \xi^-(t) < x_2^* < \xi^+(t),$$

$$\frac{\partial x_2}{\partial x_2^*} > 0, \quad \xi^+(t) < x_2^* < +\infty,$$

$$\frac{\partial x_2}{\partial x_2^*} = 0, \quad x_2^* = \xi^+(t), \; x_2^* = \xi^-(t),$$

where $\xi^\pm(t) = \pm \frac{1}{\alpha} \sqrt{\alpha w^\pi K - 1}; \; \alpha w^\pi K \geqslant 1.$

Thus the trajectories in the (t, x_2) plane will have the form shown in Fig. 2. The curve Γ_1 is the caustic and is defined by the equations

$$x_2 = u_1 (t - K) + w^s K \pm \left(\frac{1}{\alpha} \sqrt{w^\pi \alpha K - 1} - w^\pi K \operatorname{arctg} \sqrt{w^\pi \alpha K - 1}\right),$$

$$t^+ = -\tau \ln (1 - 1/\alpha w^\pi \tau), \; x_2^+ = u_1 t^+ + 1/(w^\pi \alpha)(w^s - u_1).$$

On the caustic we have the condition $\partial x_2 / \partial x_2^* = 0$ [7]; therefore, in view of (2.6) and (2.7) the quantity m_2 goes to infinity on Γ_1. The solution on the caustic can be found if we expand φ to order $(y - u_2^*)^4$ [the coefficient in front of $(y - u_2^*)^2$ is zero on Γ_1]. Substituting this expansion into (2.4), we obtain the following result on Γ_1:

$$m_2 \approx \frac{m_2^0}{\sqrt{\pi} \sigma^{1/4}} \frac{\Gamma\left(\frac{1}{4}\right)}{2^{1/4}} \frac{(\alpha x_2^*)^2 + 1}{(\alpha^3 w^\pi K^2 x_2^*)^{1/2}}, \; x_2^* = \xi^+(t). \tag{2.8}$$

Here Γ is the gamma function and $\sigma/(w^\pi)^2 \ll 1$ (from the convergence condition on the series). The solution is valid everywhere on Γ_1 except at the point (t^+, x_2^+) where $m_2 \to \infty$. Similarly, keeping terms of order $(y - u_2^*)^6$ in the expansion of φ, we find the solution at the point (t^+, x_2^+)

$$m_2^+ \approx \frac{m_2^0}{\sqrt{2\pi} \sigma^{1/3}} \frac{\Gamma\left(\frac{1}{6}\right)(18)^{1/6}}{3} \frac{1}{(\alpha^3 w^\pi K^3)^{1/3}}, \; \alpha w^\pi K = 1. \tag{2.9}$$

The applicability condition for this solution [using (2.9) and Kn >> 1] has the form

$$m_2^+ \ll 1, \; \alpha d / m_2^+ \gg 1.$$

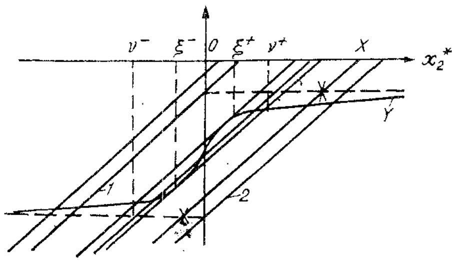

Fig. 1

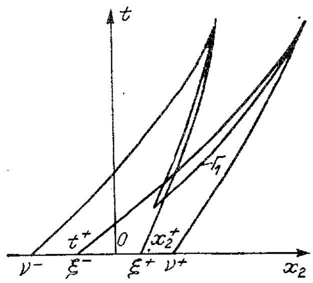

Fig. 2

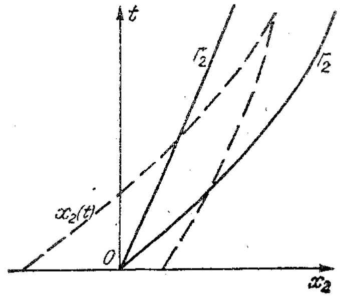

Fig. 3

The solution of the problem considered here with the initial condition $u_2 = u_2{}^0(x_2)$, $m_2 = m_2{}^0$ generally cannot be obtained in the two-fluid model [6] because inside Γ_1 three particle trajectories pass through each point. If we introduce two additional phases associated with the particles, then a solution of the type (2.6) can be obtained everywhere except on Γ_1 where m_2 goes to infinity. When the ensemble of particles is modeled as a continuous medium, only one value of the velocity is defined at each point (for a fixed phase) and, therefore, $\sigma = 0$ and it then follows from (2.8) and (2.9) that $m_2 \to \infty$ on Γ_1. We conclude that this infinity is irremovable in the framework of a collisionless continuous medium model. The singularity in m_2 can be removed by introducing collisions into the particle phase for $Kn \ll 1$ [2] or by using the kinetic equation (1.1) for $Kn \gg 1$, where $Kn \cong d/(m_2 L)$ is the Knudsen number. Using the inequality $\sigma/(w^\pi)^2 \ll 1$, $\alpha w^\pi K \sim 1$ and the solution (2.6)-(2.9), it is easy to show that the maximum value of m_2 is attained on the caustic and hence a cluster of particles forms on the caustic. The formation of the cluster results from singularities in the behavior of the particle trajectories rather than from collisions between particles, as in [2, 3].

When the width $\Delta x_2 \sim 1/\alpha$ goes to zero ($\alpha \to \infty$) and $u_2{}^0(x_2)$ transforms into a step function, it follows from (2.6) that the solution has the form

$$m_2 = \sum_{l=1}^{n} m_2^0,$$
$$x_2 = x_2^* + u_1(t-K) + u_2^0(x_2^*)K,\ u_2^0(x_2^*) = w^s - \left(\frac{2}{\pi}\right)^{-1} w^\pi \theta(x_2^*),$$
$$\theta(x_2^*) = \begin{cases} -1, & x_2^* < 0, \\ 1, & x_2^* > 0. \end{cases}$$

The function $Y = -u_2{}^0(x_2{}^*)$ is shown in Fig. 1 by dashed curves and in the region bounded by the straight lines 1 and 2 there are two roots of $Y = X$ ($n = 2$) and one root ($n = 1$) elsewhere. The corresponding pattern of trajectories is shown in Fig. 3, where the positions of the curves Γ_2 are determined by the equations

$$x_2 = u_1(t-K) + w_2 K,\ x_2 = u_1(t-K) + w_1 K.$$

In the region bounded by the curves Γ_2 we have $m_2 = 2m_2{}^0$, and elsewhere $m_2 = m_2{}^0$.

We study the stability of the stationary solution of the system (1.2)-(1.5) to small perturbations. In the one-dimensional case, neglecting the volume occupied by the particles (small m_2), the system (1.1)-(1.5) can be written in the form

$$\frac{\partial \mathcal{F}}{\partial t} + u_2 \frac{\partial \mathcal{F}}{\partial x} + \frac{\partial}{\partial u_2} F\mathcal{F} = 0,\ F = \frac{u_1 - u_2}{\tau} - g, \tag{3.1}$$
$$m_2 = \int_{-\infty}^{+\infty} \mathcal{F} du_2,\ \langle u_2 \rangle = \frac{1}{m_2} \int_{-\infty}^{+\infty} u_2 \mathcal{F} du_2,\ \rho_{22} = \mathrm{const},$$
$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} = -\frac{1}{\rho_{11}} \frac{\partial p}{\partial x} - \frac{m_2 \rho_{22}}{\rho_{11}} \frac{u_1 - \langle u_2 \rangle}{\tau} - g,$$
$$\frac{\partial \rho_{11}}{\partial t} + \frac{\partial}{\partial x}(\rho_{11} u_1) = 0,\ p = p(\rho_{11}),\ \tau = \frac{\rho_{22} d^2}{18\mu}.$$

The stationary solution of the system of equations (3.1), neglecting the compressibility of the gas, is

$$f^0 = (\rho_{11}^0; m_2^0; u_1^0; \langle u_2\rangle^0) = \text{const}, \tag{3.2}$$
$$\frac{u_1^0 - a}{\tau} = g, \frac{dp^0}{dx} = -g(\rho_{11}^0 + \rho_2^0), \ a = \langle u_2\rangle^0.$$

At the instant t = 0 a perturbation of the stationary solution is turned on:

$$f = f^0(x) + f'(x), \ f'(x) = \delta f \sin kx, \ \delta f \ll f^0.$$

Since short-wavelength perturbations are of the most interest, we consider the case of large k satisfying the inequality $k\tau c_0 \gg 1$, and for long-wavelength perturbations we obtain the inequality

$$d \ll \lambda \ll (\rho_{22}/\rho_{11})d, \tag{3.3}$$

where $\lambda = 2\pi/k$ and c_0 is the speed of sound in the gas. It follows from (3.1) that the equations for the gas phase are nonlinear and therefore we represent the solution vector for the gas phase as a sum

$$\varphi^* = \varphi^0(x) + \varphi'(x,t) \left(\varphi^0(x) = (p^0(x), u_1^0(x), \rho_{11}^0), \ \varphi'/\varphi^0 \ll 1\right), \tag{3.4}$$

and linearize the equations for the gas about the stationary solution (3.2). Taking into account (3.2)-(3.4), we obtain from the system (3.1) the following equations, all to order $O(1/k\tau c_0)$:

$$\frac{\partial \mathcal{F}}{\partial t} + u_2\frac{\partial \mathcal{F}}{\partial x} + \frac{\partial}{\partial u_2}F\mathcal{F} = 0, \ F = (a - u_2 + u_1')/\tau,$$
$$m_2 = \int_{-\infty}^{+\infty} \mathcal{F}du_2, \ \langle u_2\rangle = \frac{1}{m_2}\int_{-\infty}^{+\infty} u_2\mathcal{F}du_2, \ u_2 = a + u_2', \tag{3.5}$$

$$\frac{du_2'}{dt} = \frac{u_1^* - u_2'}{\tau}, \ \frac{dx_2}{dt} = a + u_2', \ a = \text{const},$$
$$\frac{\partial u_1'}{\partial t} + u_1^0\frac{\partial u_1'}{\partial x} + \frac{1}{\rho_{11}^0}\frac{\partial p'}{\partial x} = 0, \ p' = \rho_{11}'c_0^2,$$
$$\frac{\partial}{\partial t}\left(\frac{\rho_{11}'}{\rho_{11}^0}\right) + u_1^0\frac{\partial}{\partial x}\left(\frac{\rho_{11}'}{\rho_{11}^0}\right) + \frac{\partial u_1'}{\partial x} = 0.$$

We note that the last three equations of (3.5) describe the propagation of the perturbations in the gas and do not depend on the parameters of the second phase. This means that we can study the equations for the particles without assuming that the perturbations are small, whereas for the gas phase it is sufficient to consider the linearized equations for the perturbations. We note that u_2' is not the perturbation of u_2, but the difference of u_2 from $a \equiv \langle u_2\rangle^0$.

The initial conditions for the system (3.5) are written in the form

$$\varphi'|_{t=0} = \delta\varphi \sin kx, \tag{3.6}$$
$$\mathcal{F}|_{t=0} = \frac{m_2^\alpha}{\sqrt{2\pi\sigma}}\exp\left(-\frac{(u_2 - (a + \delta u_2 \sin kx))^2}{2\sigma}\right),$$
$$m_2^\alpha = m_2^0 + \delta m_2 \sin kx$$

where φ^0 is defined in (3.4). Rewriting the last three equations of (3.5) in characteristic form, we have, with the help of (3.6),

$$u_1' = \frac{1}{2}\left(\delta r \sin k(x - (u_1^0 + c_0)t) + \delta s \sin k(x - (u_1^0 - c_0)t)\right), \tag{3.7}$$
$$\frac{\rho_{11}'}{\rho_{11}^0} = \frac{1}{2c_0}\left(\delta r \sin k(x - (u_1^0 + c_0)t) - \delta s \sin k(x - (u_1^0 - c_0)t)\right),$$

$$\delta r = \delta u_1 + c_0 \frac{\delta\rho_{11}}{\rho_{11}^0}, \quad \delta s = \delta u_1 - c_0 \frac{\delta\rho_{11}}{\rho_{11}^0}. \tag{3.7}$$

Integrating the equation of motion of a particle, we find

$$u_2' = u_2'^* e^{-t/\tau} + e^{-t/\tau}\left(\int_0^t \frac{u_1'(t', x_2(t'))}{\tau} e^{t'/\tau} dt'\right), \tag{3.8}$$

$$x_2 = x_2^* + at + u_2'^* K(t) + \int_0^t dt' e^{-t'/\tau} \int_0^{t'} \frac{u_1'(t'', x_2(t''))}{\tau} e^{t''/\tau} dt'',$$

where $K(t) = \tau(1 - e^{-t/\tau})$; $u_1'(t'', x)$ is determined by (3.7). The solution of the second equation of (3.8) is found by iteration. We choose as the zeroth approximation

$$x_2^{(0)} = x_2^* + at + u_2'^* K(t). \tag{3.9}$$

Substituting $x_2^{(0)}$ into the first equation of (3.8) and the result into the second equation, we compute $x_2^{(1)}$, then $x_2^{(2)}$, and so on, by repeating this procedure. Assuming that $x_2^{(n)} = x_2^* + at + \varepsilon K$, in order to calculate $x_2^{(n+1)}$ we must evaluate an integral of the form

$$I = \int_0^t dt' e^{-t'/\tau} \int_0^{t'} \frac{e^{t''/\tau}}{\tau}\left[\frac{\delta s}{2}\sin\left(kb^- t'' + k\varepsilon K(t'') + \alpha\right) - \frac{\delta r}{2}\sin\left(kb^+ t'' - k\varepsilon K(t'') - \alpha\right)\right] dt'',$$

where $b^- = c_0 - (u_1^0 - a)$; $\alpha = kx_2^*$; $kb^-\tau \gg 1$ (large k); $b^+ = c_0 + (u_1^0 - a)$; $kb^+\tau \gg 1$; $b^- > 0$ (subsonic with respect to the flow velocity). Using the properties of integrals of rapidly oscillating functions [8], we have

$$I = (A - \varepsilon B) K \cos\alpha + O(1/(\omega^+)^2 + 1/(\omega^-)^2), \tag{3.10}$$

where $\omega^- = kb^-\tau$; $\omega^+ = kb^+\tau$;

$$A = \frac{\delta s}{2\omega^-} - \frac{\delta r}{2\omega^+}; \quad B = \frac{\delta s}{(2\omega^- b^-)} + \frac{\delta r}{(2\omega^+ b^+)}.$$

Taking into account (3.8) and (3.10), we obtain the n-th iteration

$$x_2^{(n)} = x_2^{(0)} + K\left(\frac{A}{B}\sum_{k=1}^{n}(-1)^{(k-1)}\Delta^k + u_2'^*\sum_{k=1}^{n}(-1)^k\Delta^k\right), \tag{3.11}$$

where $\Delta = B\cos kx_2^*$. Using mathematical induction, it is a simple matter to prove that (3.11) is the correct solution and the sequence $x_2^{(0)}, x_2^{(1)}, \ldots, x_2^{(n)}$ is a geometric progression with a sum equal to

$$x_2 = x_2^{(0)} + K\left(\frac{A}{B} - u_2'^*\right)\frac{\Delta}{1+\Delta}. \tag{3.12}$$

Because the integral (3.10) was calculated to within an accuracy of $O(1/(\omega^+)^2 + 1/(\omega^-)^2)$, the final expression for x_2, to the same accuracy, can be written [with the help of (3.12)] in the form

$$x_2 = x_2^* + at + u_2'^* K + \left(\frac{\delta s}{2\omega^-} - \frac{\delta r}{2\omega^+}\right) K\cos kx_2^* - \left(\frac{\delta s}{2\omega^- b^-} + \frac{\delta r}{2\omega^+ b^+}\right) u_2'^* K \cos kx_2^*. \tag{3.13}$$

Equation (3.13) can be obtained from (3.11) if we put n = 1, i.e., when it is sufficient to use a single iteration.

Substituting (3.7) and (3.13) into the first equation of (3.8), we have

$$u_2' = u_2'^* e^{-t/\tau} + e^{-t/\tau}\left(\frac{\delta s}{2} I_1 - \frac{\delta r}{2} I_2\right) + O\left((\delta w)^2, \frac{u_2'}{c_0}\delta w\right), \tag{3.14}$$

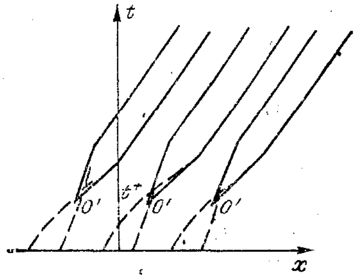

Fig. 4

$$I_{1,2}=\frac{1}{\omega^{\pm}}\left[\left(1\pm\frac{u_2'^*}{b^{\pm}}\right)\cos kx_2^*-\left(e^{t/\tau}\pm\frac{u_2'^*}{b^{\pm}}\right)\cos\left(\omega^{\pm}\varphi^{\pm}-kx_2^*\right)\right]+O(1/\omega^{\pm})^2, \tag{3.14}$$

$$\varphi^{\pm}=t/\tau\mp\frac{u_2'^*}{b^{\pm}\tau}K\mp\frac{\delta\tilde{u}_1}{b^{\pm}\tau}K,\ \delta\tilde{u}_1=\delta u_1\cos kx_2^*,$$

and from this result we can compute the partial derivatives $\left(\frac{\partial x_2^*}{\partial u_2'^*}\right), \left(\frac{\partial u_2'}{\partial u_2'^*}\right)$ at fixed t and x_2 in the form

$$\frac{\partial x_2^*}{\partial u_2'^*}=-K\Big/\left(1-\delta w\frac{K}{\tau}\sin kx_2^*\right)+O\left(\delta w\frac{u_2'^*}{c_0},\frac{\delta w}{\omega^0}\right), \tag{3.15}$$

$$\frac{\partial u_2'}{\partial u_2'^*}=e^{-t/\tau}+O\left(e^{-t/\tau}\frac{\delta u_1}{c_0\omega^0}\right),$$

$$\delta w=\frac{1}{2}\left(\frac{\delta s}{b^-}-\frac{\delta r}{b^+}\right),\ \omega^0=kc_0\tau.$$

As follows from (3.1), in order to determine m_2 and $\langle u_2\rangle$ it is necessary to evaluate the integral

$$\langle\Phi\rangle=\frac{m_2^{\alpha}}{m_2}\frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{+\infty}\Phi(u_2)\,e^{t/\tau}e^{-\frac{\left(u_2'^*-\delta u_2\sin kx_2^*\right)^2}{2\sigma}}du_2. \tag{3.16}$$

Substituting (3.13) and (3.15) into (3.16) and changing variables:
and using the inequality $\sqrt{\sigma}/c_0 \ll 1$, we have $du_2=du_2',\ du_2'=\frac{\partial u_2'}{\partial u_2'^*}\partial u_2'^*$,

$$\langle\Phi\rangle=\frac{m_2^{\alpha}}{m_2}\frac{1}{\sqrt{2\pi\sigma}}\int_{-\infty}^{+\infty}\Phi(u_2)\,e^{-\frac{\left(u_2'^*-\delta u_2\sin kx_2^*\right)^2}{2\sigma}}du_2'^*; \tag{3.17}$$

$$x=x_2^*+at+u_2'^*K+\frac{\delta w}{k\tau}K\cos kx_2^*. \tag{3.18}$$

Expanding the argument of the exponential about its maximum, which is determined from the equation

$$y-\delta u_2\sin kx_2^*(y,t,x)=0, \tag{3.19}$$

we find

$$f=\left(1+\delta u_2kK\cos kx_2^*(y,t,x)\right)^2\left(u_2'^*-y\right)^2, \tag{3.20}$$

where f is the argument of the exponential in (3.17): $f=(u_2'^*-\delta u_2\sin k x_2^*)^2$. Substituting (3.20) into (3.17) we obtain

$$m_2\langle\Phi\rangle=\sum_{i=1}^{n}\frac{m_2^{\alpha}\Phi\left(y_i,x_{2i}^*\right)}{\left|1+\delta u_2kK\cos kx_2^*(y_i,t,x)\right|}, \tag{3.21}$$

where $\Phi = 1$, $\Phi = u_2$, and n is the number of roots of Eqs. (3.18) and (3.19). Putting $\Phi = 1$, $n = 1$ into (3.21) and comparing with the corresponding expression for m_2 obtained in [2] with the two-fluid model, we find that they are identical.

If $\delta u_2 k\tau < 1$, then m_2 is finite everywhere in the half-plane $t \geqq 0$, $-\infty < x < +\infty$ and in the opposite case ($\delta u_2 k\tau \geqq 1$) there exist points where $m_2 \to \infty$. Setting the denominator in (3.21) equal to zero, we find an equation for a curve in the t, x_2^* plane on which m_2 is infinite:

$$1 + \delta u_2 kK \cos kx_2^* = 0. \tag{3.22}$$

Expanding (3.22) we have

$$x_2^* = \frac{\pi}{k} + \frac{2\pi m}{k} \pm \frac{1}{k}\left(\arccos\frac{1}{\delta u_2 kK}\right), \quad m = 0, \pm 1, \ldots . \tag{3.23}$$

Substituting (3.23) into (3.18), we can write

$$x = \frac{\pi}{k} + \frac{2\pi m}{k} \pm \frac{1}{k}\left(\arccos\frac{1}{\delta u_2 kK} - \sqrt{(\delta u_2 kK)^2 - 1}\right) - \delta w/(\delta u_2 k^2\tau) + at \quad (t > -\tau\ln(1 - 1/(\delta u_2 k\tau))). \tag{3.24}$$

Using (3.17), (3.18), and (3.22), we find that on the curve defined by (3.24) to within terms of order $O(\delta w)$ we have the identity $\partial x/\partial x_2^* = 0$; these curves are called caustics [7]. The distribution of trajectories (dashed curves) and caustics (solid curves) are shown for this case in Fig. 4, where $t^+ = -\tau\ln(1 - 1/\delta u_2 k\tau)$. The calculation of m_2 on the caustics is carried out by expanding f up to terms of the fourth power $(u_2'^* - y)^4$:

$$f \approx \frac{1}{4}(kK)^2((\delta u_2 kK)^2 - 1)(u_2'^* - y)^4,$$

and substituting the result into (3.17). We thereby find m_2 on a caustic in the form

$$m_2 \approx \frac{2m_2^\alpha \Gamma\left(\frac{1}{4}\right)}{\sqrt{2\pi}\sigma^{1/4}\left(2(kK)^2\left((\delta u_2 kK)^2 - 1\right)\right)^{1/4}}, \tag{3.25}$$

where Γ is the gamma function. This formula is valid everywhere on the caustic except the point O' (and its neighborhood). At the point O' we have the expansion $f \approx (1/36)(\delta u_2(kK)^3)^2 \cdot (u_2'^* - y)^6$ and, therefore, m_2 is given by the formula

$$m_2 \approx \frac{m_2^\alpha}{3\sigma^{1/3}}\sqrt{\frac{2}{\pi}}\,\Gamma\left(\frac{1}{6}\right)\Big/\left(Kk(\delta u_2)^{1/3}\right). \tag{3.26}$$

The applicability condition of (3.26) has the form

$$m_2' \ll 1, \; kd/m_2' \gg 1, \; kd < 1,$$

where $m_2' \approx m_2^0(\delta u_2^2/\sigma)^{1/3}$; $\sigma/\delta u_2^2 < 1$; $m_2^0 \ll 1$.

Therefore, we have shown that in the continuum-discrete model, without taking into account the volume of the particles, a small perturbation arising at $t = 0$ on $-\infty < x < +\infty$ remains finite everywhere in the half-plane $t > 0$, $-\infty < x < +\infty$. The maximum amplitude of the perturbation occurs on the caustics defined by (3.24) and it is inversely proportional to the width of the distribution function raised to a fractional power [Eqs. (3.25) and (3.26)], whereas in the two-fluid model (also without taking into account the volume of the particles) a small perturbation diverges on the caustics according to the law (3.21).

LITERATURE CITED

1. L. A. Klebanov, A. E. Kroshilin, et al., "On the hyperbolicity, stability, and correctness of the Cauchy problem for the system of equations of the two-velocity motion of a two-fluid medium," Prikl. Mat. Mekh., 46, No. 1 (1982).
2. A. N. Kraiko, "On the correctness of the Cauchy problem for the two-fluid model of the flow of a gas mixture with particles," Prikl. Mat. Mekh., 46, No. 3 (1982).
3. A. N. Kraiko, "Theory of the two-fluid flow of a gas and dispersed particles," in: Hydrodynamics and Heat Exchange in a Two-Phase Medium [in Russian], Proc. II All-Union School on Thermal Physics, Novosibirsk (1981).

4. S. V. Iordanskii and A. G. Kulikovskii, "On the motion of a fluid containing small particles," Izv. Akad. Nauk, Mekh. Zhidk. Gaza, No. 4 (1977).
5. V. I. Myasnikov, "Statistical model of the mechanical behavior of dispersed systems," in: Mechanics of Multicomponent Media in Technological Processes [in Russian], Nauka, Moscow (1978).
6. A. N. Kraiko and L. E. Sternin, "Theory of the flow of a two-velocity continuous medium with solid or liquid particles," Prikl. Mat. Mekh., 29, No. 3 (1965).
7. Ya. B. Zel'dovich and A. D. Myshkis, Elements of Mathematical Physics [in Russian], Nauka, Moscow (1973).
8. Ya. B. Zel'dovich and A. D. Myshkis, Elements of Applied Mathematics [in Russian], Nauka, Moscow (1972).

CALCULATION OF THE NONEQUILIBRIUM PARAMETERS OF AIR AT THE SURFACES OF MODELS AND IN THE WAKES BEHIND THEM FOR THE CONDITIONS OF AEROBALLISTIC EXPERIMENTS

I. G. Eremeitsev and N. N. Pilyugin

UDC 629.7.018.3

The calculation of the nonequilibrium, quasi-one-dimensional flow of chemically reactive gas mixtures is of practical interest in connection with the study of relaxation processes, obtaining gasdynamic jets for physical measurements, and the investigation of plasma supersonic phenomena in the wake behind a body, etc.

Calculations of chemically nonequilibrium, supersonic, quasi-one-dimensional flows are presented in [1-8] and elsewhere. Here various algorithms are used to solve such problems for flows in nozzles and stream tubes near a body. At present the fields of nonequilibrium parameters at the surfaces of spherically blunted cones are calculated for certain conditions of streamline flow using stream tubes, while calculated results for inviscid flow in wakes are absent. In expansion behind the stern cut of a body, where the gas temperature is sharply reduced, it is necessary to make additional allowance for important reactions with the participation of electrons, negative ions, and polyatomic molecules. Calculations of nonequilibrium parameters in the flow over bodies with surfaces of other shapes, in a wide range of variation of the initial parameters, are also necessary for the comparison and treatment of the results of aeroballistic experiments. However, the absence of calculation methods that are convenient and rapid for execution on computers has prevented making such comparative investigations and giving practical recommendations up to now.

The problem of the flow of a chemically nonequilibrium, partially ionized, multicomponent, inviscid gas from a spherical supersonic source was studied in detail in [9]; from the calculations it is seen that in a number of important cases one can use a constant value of the effective adiabatic index, making it possible to obtain a one-to-one connection between the area of a stream tube and the gas pressure.

In the present paper we give a single algorithm for the computer calculation of the direct and inverse quasi-one-dimensional problems of the flow of chemically nonequilibrium, multicomponent air. The formulation and ways of solving a number of problems of nonequilibrium aerodynamics are discussed on the basis of the calculation method developed.

1. Let us consider the steady quasi-one-dimensional flow of a chemically nonequilibrium gas. The system of dimensionless equations describing such flow has the form [1]

$$\rho v S(x) = 1, \quad \rho v \frac{dv}{dx} = -\frac{dp}{dx}, \tag{1.1}$$

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 101-111, March-April, 1986. Original article submitted February 15, 1985.